

AN EXPLICIT INTEGRAL REPRESENTATION OF SIEGEL-WHITTAKER FUNCTIONS ON $\mathrm{Sp}(2, \mathbb{R})$ FOR THE LARGE DISCRETE SERIES REPRESENTATIONS

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In Memoriam: Fumiyuki Momose

ABSTRACT. We obtain an explicit integral representation of Siegel-Whittaker functions on $\mathrm{Sp}(2, \mathbb{R})$ for the large discrete series representations. We have another integral expression different from that of Miyazaki [7].

1. INTRODUCTION

In this article, we study Siegel-Whittaker functions on $G = \mathrm{Sp}(2, \mathbb{R})$, the real symplectic group of degree two for the large discrete series representations. Let P_S be the Siegel parabolic subgroup of G , which is a maximal parabolic subgroup with abelian unipotent radical N_S . Let π be an admissible representation of G and ξ be a *definite* unitary character of N_S . Siegel-Whittaker model for an admissible π is a realization of π in the induced module from a certain closed subgroup R which contains N_S . (See (2.2) for definition of R .) We consider the intertwining space

$$SW(\pi; \eta) = \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi, C_{\eta}^{\infty}(R \backslash G)),$$

where η is an irreducible R -module such that $\eta|_{N_S}$ contains ξ , K is a maximal compact subgroup of G and $\mathfrak{g}_{\mathbb{C}}$ is the complexification of the Lie algebra of G . A function in this realization is called a *Siegel-Whittaker function* for π .

Takuya Miyazaki obtained a system of partial differential equations satisfied by Siegel-Whittaker functions for the large discrete representations in [7]. He also obtained multiplicity one property and its formal power series solutions. In this article, we investigate further the system obtained by Miyazaki (Proposition 3.1) and give an explicit integral representation of the Siegel-Whittaker functions, which is of rapid decay, for a large discrete series representation π . (Theorem 5.1.) In other words, we show that the rapidly decreasing Siegel-Whittaker function for the large discrete series representation π is uniquely determined and (up to polynomials) described by the *partially confluent hypergeometric*

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functions in the A -radial part $(a_1, a_2) \in (\mathbb{R}_{>0})^2$:

$$(1.1) \quad \Psi_{\alpha,\beta}(a_1, a_2) = e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} \int_0^1 F_{SW}(2\pi h_1 a_1^2 t + 2\pi h_2 a_2^2 (1-t)) t^{\alpha-1} (1-t)^{\beta-1} dt,$$

with

$$F_{SW}(x) = F(x) = e^x x^{-\gamma} W_{\kappa,\mu}(2x).$$

Here, $W_{\kappa,\mu}(x)$ is the Whittaker's classical confluent hypergeometric function, $\alpha, \beta, \gamma > 0$ and κ, μ are half integers depending on parameters of representations π and η . This kind of integral already appeared in Gon [1], treated the cases for the large discrete series representations and P_J -principal series representations of $SU(2, 2)$.

The reason we begin to believe the validity of similar formula for $Sp(2, \mathbb{R})$, is the paper Hirano-Ishii-Oda [2]. By this new integral expression (1.1), it seems possible to extend the former argument in [2] of the confluence from Siegel-Whittaker functions to Whittaker functions for P_J -principal series, to the confluence from Siegel-Whittaker functions to Whittaker functions [8] for the large discrete series of $Sp(2, \mathbb{R})$. Here we recall the template of the integral formulas of Whittaker functions in $(a_1, a_2) \in (\mathbb{R}_{>0})^2$:

$$(1.2) \quad \mu_W(a_1, a_2) \int_0^\infty F_W(t) t^{-C} e^{-At^2/a_2^2 - B a_1^2/t^2} dt,$$

where $F_W(t) = W_{0,\gamma}(t)$ with a positive integral parameter γ , $\mu_W(a_1, a_2)$ is a monomial in a_1, a_2 times $e^{\delta a_2^2}$ ($\delta > 0$) and A, B, C are positive real parameters.

Probably we can proceed a bit more. Since Iida [3], Theorems 8.7 and 8.9 (Formulas (8.8) and (8.10), resp.) give analogous integral expressions for the matrix coefficient of P_J -principal series, and also Oda [9] (p.247), Theorem 6.2 similarly gives analogous integral expression for matrix coefficients of the large discrete series. We notice that the template of formulas in Theorem 6.2 of [9] is given by

$$(1.3) \quad \mu_{MC}(a_1, a_2) \int_0^1 F_{MC}(tx_1 + (1-t)x_2) t^{\alpha-1} (1-t)^{\beta-1} dt,$$

with $x_i = -(a_i - a_i^{-1})^2/4$ ($i = 1, 2$). Here $F_{MC}(x) = {}_2F_1(A, B; C : x)$ with A, B, C, α, β determined by Harish-Chandra parameters and the components of the minimal K -type. The elementary factor $\mu_{MC}(a_1, a_2)$ is a monomial of $((a_i \pm a_i^{-1})/2)^{\pm 1/2}$ ($i = 1, 2$).

These facts suggests that there will be a deformation of confluence (1.3) \mapsto (1.1) from the matrix coefficients realization to Siegel-Whittaker realization in a very simple natural way. We believe the similar argument is possible for the principal series: but in this case the natures of integrands of Whittaker case Ishii [5], Theorem 3.2 and Siegel-Whittaker case [4], Theorem 10.1 are still difficult to deform.

For the P_J -principal series of the other group $SU(2, 2)$, we have explicit integral expression of the matrix coefficients (Theorem 5.4 of [6]) similar to the above formula (1.3). We can expect the confluence from this formula to the integral expression of the Siegel-Whittaker realization of [1] analogous to (1.1), corresponding to the limit of one parameter

conjugations of a compact subgroup K of $\mathrm{SU}(2, 2)$:

$$\lim_{t \rightarrow 0} h_t K h_t^{-1} = R.$$

The deformation of the Siegel-Whittaker function of the P_J -principal series to the Whittaker function could be handled in a similar way as in [2].

In view of this observation, we may hope that similar phenomenon occurs for more general groups $\mathrm{SO}(2, q)$ ($q > 4$).

2. PRELIMINARIES

2.1. Basic notations. Let G be the real symplectic group of degree two:

$$\mathrm{Sp}(2, \mathbb{R}) = \left\{ g \in \mathrm{SL}(4, \mathbb{R}) \mid {}^t g J_2 g = J_2 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\},$$

with 1_2 the unit matrix of degree two and 0_2 the zero matrix of degree two.

Fix a maximal compact subgroup K of G by

$$K = \left\{ k(A, B) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \mid A, B \in \mathrm{M}(2, \mathbb{R}) \right\}.$$

It is isomorphic to the unitary group $\mathrm{U}(2)$ via the homomorphism

$$K \ni k(A, B) \mapsto A + \sqrt{-1}B \in \mathrm{U}(2).$$

We define a certain spherical subgroup R of G as follows. Let $P_S = L_S \ltimes N_S$ be the Siegel parabolic subgroup with the Levi part L_S and the abelian unipotent radical N_S given by

$$L_S = \left\{ \begin{pmatrix} A & 0_2 \\ 0_2 & {}^t A^{-1} \end{pmatrix} \mid A \in \mathrm{GL}(2, \mathbb{R}) \right\},$$

$$N_S = \left\{ n(T) = \begin{pmatrix} 1_2 & T \\ 0_2 & 1_2 \end{pmatrix} \mid {}^t T = T \in \mathrm{M}(2, \mathbb{R}) \right\}.$$

Fix a non-degenerate unitary character ξ of N_S by

$$\xi(n(T)) = \exp(2\pi\sqrt{-1}\mathrm{tr}(H_\xi T))$$

with $H_\xi = \begin{pmatrix} h_1 & h_3/2 \\ h_3/2 & h_2 \end{pmatrix} \in \mathrm{M}(2, \mathbb{R})$ and $\det H_\xi \neq 0$. Consider the action of L_S on N_S by conjugation and the induced action on the character group $\widehat{N_S}$. Define $\mathrm{SO}(\xi)$ to be the identity component of the subgroup of L_S which stabilize ξ :

$$\mathrm{SO}(\xi) := \mathrm{Stab}_{L_S}(\xi)^\circ = \left\{ \begin{pmatrix} A & 0_2 \\ 0_2 & {}^t A^{-1} \end{pmatrix} \in L_S \mid {}^t A H_\xi A = H_\xi \right\}.$$

Then $\mathrm{SO}(\xi)$ is isomorphic to $\mathrm{SO}(2)$ if $\det H_\xi > 0$ and to $\mathrm{SO}_o(1, 1)$ if $\det H_\xi < 0$. In this article we treat the case that ξ is a ‘definite’ character, that is $\det H_\xi > 0$. So we may assume $h_1, h_2 > 0$ and $h_3 = 0$ without loss of generality. We sometimes identify the element

of $\mathrm{SO}(\xi)$ with its upper left 2×2 component. Fix a unitary character χ_{m_0} ($m_0 \in \mathbb{Z}$) of $\mathrm{SO}(\xi) \cong \mathrm{SO}(2)$ by

$$(2.1) \quad \chi_{m_0} \left(\begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix}^{-1} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix} \right) = \exp(\sqrt{-1} m_0 \theta).$$

We define

$$(2.2) \quad R = \mathrm{SO}(\xi) \ltimes N_S \quad \text{and} \quad \eta = \chi_{m_0} \boxtimes \xi.$$

Taking a maximal split torus A of G by

$$A = \{a = (a_1, a_2) = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1, a_2 > 0\},$$

we have the decomposition $G = RAK$.

2.2. Siegel-Whittaker functions. We consider the space $C_\eta^\infty(R \backslash G)$ of complex valued C^∞ functions f on G satisfying

$$f(rg) = \eta(r)f(g) \quad \forall (r, g) \in R \times G.$$

By the right translation, $C_\eta^\infty(R \backslash G)$ is a smooth G -module and we denote the same symbol its underlying $(\mathfrak{g}_\mathbb{C}, K)$ -module. For an irreducible admissible representation (π, H_π) of G and the subspace $H_{\pi, K}$ of K -finite vectors, the intertwining space

$$\mathcal{I}_{\eta, \pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(H_{\pi, K}, C_\eta^\infty(R \backslash G))$$

between the $(\mathfrak{g}_\mathbb{C}, K)$ -modules is called the space of *algebraic Siegel-Whittaker functionals*. For a finite-dimensional K -module (τ, V_τ) , denote by $C_{\eta, \tau}^\infty(R \backslash G/K)$ the space

$$\{\phi: G \rightarrow V_\tau, C^\infty \mid \phi(rgk) = \eta(r)\tau(k^{-1})\phi(g) \quad \forall (r, g, k) \in R \times G \times K\}.$$

Let (τ^*, V_{τ^*}) be a K -type of π and $\iota: V_{\tau^*} \rightarrow H_\pi$ be an injection. Here, τ^* means the contragredient representation of τ . Then for $\Phi \in \mathcal{I}_{\eta, \pi}$, we can find an element ϕ_ι in

$$C_{\eta, \tau}^\infty(R \backslash G/K) = C_\eta^\infty(R \backslash G) \otimes V_{\tau^*} \cong \mathrm{Hom}_K(V_{\tau^*}, C_\eta^\infty(R \backslash G))$$

via $\Phi(\iota(v^*))(g) = \langle v^*, \phi_\iota(g) \rangle$ with $\langle \cdot, \cdot \rangle$ the canonical pairing on $V_{\tau^*} \times V_\tau$.

Since there is the decomposition $G = RAK$, our generalized spherical function ϕ_ι is determined by its restriction $\phi_\iota|_A$, which we call the *radial part* of ϕ_ι . For a subspace X of $C_{\eta, \tau}^\infty(R \backslash G/K)$, we denote $X|_A = \{\phi|_A \in C^\infty(A) \mid \phi \in X\}$.

Let us define the space $\mathrm{SW}(\pi, \eta, \tau)$ of Siegel-Whittaker functions and its subspace $\mathrm{SW}(\pi, \eta, \tau)^{\mathrm{rap}}$ as follows:

$$\mathrm{SW}(\pi, \eta, \tau) = \bigcup_{\iota \in \mathrm{Hom}_K(\tau^*, \pi)} \{\phi_\iota \mid \Phi \in \mathcal{I}_{\eta, \pi}\}$$

and

$$\mathrm{SW}(\pi, \eta, \tau)^{\mathrm{rap}} = \{\phi_\iota \in \mathrm{SW}(\pi, \eta, \tau) \mid \phi_\iota|_A \text{ decays rapidly as } a_1, a_2 \rightarrow \infty\}.$$

We call an element in $\mathrm{SW}(\pi, \eta, \tau)$ a *Siegel-Whittaker function* for (π, η, τ) .

2.3. Parametrization of the discrete series representations. Let $E_{ij} \in M_4(\mathbb{R})$ be the matrix unit with 1 as its (i, j) -component and 0 at the other entries. The root system of G with respect to a compact Cartan subgroup

$$T = \exp(\mathbb{R}(E_{13} - E_{31}) + \mathbb{R}(E_{24} - E_{42}))$$

is given by a set of vectors in the Euclidean plane:

$$\{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}.$$

Here,

$$\begin{aligned} \varepsilon_1(r_1(E_{13} - E_{31}) + r_2(E_{24} - E_{42})) &= \sqrt{-1}r_1, \\ \varepsilon_2(r_1(E_{13} - E_{31}) + r_2(E_{24} - E_{42})) &= \sqrt{-1}r_2. \end{aligned}$$

We fix a subset of simple roots and the associated positive roots by

$$\{\varepsilon_1 - \varepsilon_2, \varepsilon_2\}, \quad \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, 2\varepsilon_2\}$$

respectively.

Then the set of the unitary characters of T (or their derivatives) is identified naturally with $\mathbb{Z} \oplus \mathbb{Z}$, and the subset consisting of dominant integral weight is

$$\Xi = \{(n_1, n_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid n_1 \geq n_2\}.$$

There is a bijection between \widehat{K} and Ξ by highest weight theory. Because the half-sum of the positive root is integral, the discrete series representations of $G = \mathrm{Sp}(2, \mathbb{R})$ are parametrized by the subset of regular elements in Ξ :

$$\Xi' = \{(n_1, n_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid n_1 > n_2, n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0\}.$$

Here the condition $n_1 > n_2$ means the positivity of weight (n_1, n_2) with respect to the compact root $\varepsilon_1 - \varepsilon_2 = (1, -1)$.

The subsets $\Xi_I = \{(n_1, n_2) \mid n_1 > n_2 > 0\}$ and $\Xi_{IV} = \{(n_1, n_2) \mid 0 > n_1 > n_2\}$ parametrize the holomorphic discrete series and the anti-holomorphic discrete series representations, respectively. Set

$$\Xi_{II} = \{(n_1, n_2) \mid n_1 > 0 > n_2, n_1 + n_2 > 0\},$$

and

$$\Xi_{III} = \{(n_1, n_2) \mid n_1 > 0 > n_2, 0 > n_1 + n_2\}.$$

Then the union $\Xi_{II} \cup \Xi_{III}$ parametrize the large discrete series representations of G .

3. MIYAZAKI'S RESULTS

Miyazaki derived a system of partial differential equations satisfied by Siegel-Whittaker functions for the large discrete representations in [7]. He also obtained multiplicity one property. We recall his results in this section.

Let $\tau = \tau_{(\lambda_1, \lambda_2)} = \mathrm{Sym}^{\lambda_1 - \lambda_2} \otimes \det^{\lambda_2}$ be the irreducible K -module with the highest weight (λ_1, λ_2) , then the dimension of τ is $d + 1$ with $d = \lambda_1 - \lambda_2$. We take the basis $\{v_j\}_{j=0}^d$ of V_{τ^*} with $\tau^* = \tau_{(-\lambda_2, -\lambda_1)}$ as in [7, Lemma 3.1].

We remark on a compatibility condition. For a non-zero function ϕ in $C_{\eta, \tau^*}^\infty(R \backslash G/K)$, we have

$$\phi(a) = \phi(mam^{-1}) = (\chi_{m_0} \boxtimes \xi)(m) \tau_{(-\lambda_2, -\lambda_1)}(m) \phi(a),$$

where, $a \in A$ and $m \in \mathrm{SO}(\xi) \cap Z_K(A) = \{\pm 1_4\}$. If we take $m = -1_4$, $(\chi_{m_0} \boxtimes \xi)(m) = \chi_{m_0}(m) = (-1)^{m_0}$ and $\tau_{(-\lambda_2, -\lambda_1)}(m) = (-1)^d$ imply that $(m_0 + d)/2$ is an integer.

Proposition 3.1 (Miyazaki [7]). *Let $\pi = \pi_\Lambda$ be a large discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\lambda_1 - 1, \lambda_2) \in \Xi_\Pi$ and its minimal K -type $\tau = \tau_{(\lambda_1, \lambda_2)}$. Let ξ be a unitary character of N_S associated with a positive definite matrix $H_\xi = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$. Put $\eta = \chi_{m_0} \boxtimes \xi$, as in (2.1). Then we have the following:*

(i) *we have $\dim_{\mathbb{C}} \mathrm{SW}(\pi, \eta, \tau) \leq 4$ and a function*

$$\phi_{\mathrm{SW}}(a) = \sum_{j=0}^d \left\{ (\sqrt{h_1} a_1)^{\lambda_1 - j} (\sqrt{h_2} a_2)^{\lambda_2 + j} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} c_j(a) \right\} v_j$$

is in the space $\mathrm{SW}(\pi_\Lambda, \eta, \tau)|_A$ if and only if $\{c_j(a)\}_{j=0}^d$ is a smooth solution of the following system:

$$(3.1) \quad \left[\partial_1 + j \frac{h_2 a_2^2}{\Delta} \right] c_{j-1}(a) + \sqrt{-1} m_0 \frac{h_2 a_2^2}{\Delta} c_j(a) - (d - j) \frac{h_2 a_2^2}{\Delta} c_{j+1}(a) = 0 \quad (1 \leq j \leq d),$$

$$(3.2) \quad j \frac{h_1 a_1^2}{\Delta} c_{j-1}(a) + \sqrt{-1} m_0 \frac{h_1 a_1^2}{\Delta} c_j(a) + \left[\partial_2 - (d - j) \frac{h_1 a_1^2}{\Delta} \right] c_{j+1}(a) = 0 \quad (0 \leq j \leq d - 1),$$

$$(3.3) \quad h_1 a_1^2 \left[\partial_2 - 8\pi h_2 a_2^2 - 2j \frac{h_2 a_2^2}{\Delta} + 2\lambda_2 - 2 \right] c_{j-1}(a) - 2\sqrt{-1} m_0 \frac{h_1 a_1^2 h_2 a_2^2}{\Delta} c_j(a) \\ + h_2 a_2^2 \left[\partial_1 - 8\pi h_1 a_1^2 + 2(d - j) \frac{h_1 a_1^2}{\Delta} + 2\lambda_2 - 2 \right] c_{j+1}(a) = 0 \quad (1 \leq j \leq d - 1),$$

with $\partial_i = a_i(\partial/\partial a_i)$ ($i = 1, 2$) and $\Delta = h_1 a_1^2 - h_2 a_2^2$.

(ii) $\dim_{\mathbb{C}} \mathrm{SW}(\pi, \eta, \tau)^{\mathrm{rap}} \leq 1$.

This is a paraphrase of Propositions 10.2, 10.7 and Theorem 11.5 of [7]. Here (3.1), (3.2), (3.3) are essentially identical equations to (10.4), (10.5), (10.6) of [7] deduced from Proposition 10.2. However we replace the symbol $\chi(Y_\eta)$ of [7] by its explicit value $\sqrt{-1} m_0 / \sqrt{h_1 h_2}$, and the symbol D of [7] by Δ .

4. PARTIALLY CONFLUENT HYPERGEOMETRIC FUNCTIONS IN TWO VARIABLES

We introduce and study certain partially confluent hypergeometric functions in two variables on $A \simeq (\mathbb{R}_{>0})^2$. These functions play a key role in describing Siegel-Whittaker functions $\phi_{\mathrm{SW}}(a)$. We remark that these types of confluent hypergeometric functions have also appeared in [1], for the large discrete series representations and P_J -principal series representations of $\mathrm{SU}(2, 2)$, and [2], for the P_J -principal series representations of $\mathrm{Sp}(2, \mathbb{R})$.

Definition 4.1 (Partially confluent hypergeometric functions). Let $\pi = \pi_\Lambda$ be a large discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\lambda_1 - 1, \lambda_2) \in \Xi_\Pi$ and its minimal K -type $\tau = \tau_{(\lambda_1, \lambda_2)}$. Let ξ be a unitary character of N_S associated with a positive definite matrix $H_\xi = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$. Put $\eta = \chi_{m_0} \boxtimes \xi$, as in (2.1).

We assume that

$$|m_0| \geq d \quad \text{and} \quad |m_0| \equiv d \pmod{2}.$$

For $0 \leq k \leq d$, define

$$(4.1) \quad \begin{aligned} f_k(a) &= f_k(a; [\pi, \eta, \tau]) \\ &= (\sqrt{h_1}a_1)^{2k+1} (\sqrt{h_2}a_2)^{2d+1-2k} (h_1a_1^2 - h_2a_2^2)^{\frac{|m_0|-d}{2}} \\ &\quad \times \int_0^1 F(2\pi h_1a_1^2t + 2\pi h_2a_2^2(1-t)) t^{\frac{|m_0|-d-1}{2}+k} (1-t)^{\frac{|m_0|+d-1}{2}-k} dt, \end{aligned}$$

with

$$F(x) = e^x x^{-\frac{|m_0|+\lambda_1+1}{2}} W_{\frac{\lambda_1-|m_0|-1}{2}, \frac{\lambda_2}{2}}(2x).$$

Here, $W_{\kappa, \mu}(z)$ is Whittaker's confluent hypergeometric function. (See [10] Chapter 16 for definition.)

Since the indices $\alpha_k - 1 = \frac{|m_0|-d-1}{2} + k$, $\beta_k - 1 = \frac{|m_0|+d-1}{2} - k$ in the integrand of f_k satisfy $\alpha_k, \beta_k > 0$ ($0 \leq k \leq d$), we see that $f_k(a)$ is a smooth function on A and of moderate growth when each a_1, a_2 tends to infinity. We have further more,

Proposition 4.2. *Partially confluent hypergeometric functions $\{f_k(a)\}_{k=0}^d$ satisfy the following system of the difference-differential equations.*

$$(4.2) \quad \partial_1 f_k = -(2k+1) \frac{h_2a_2^2}{\Delta} f_k + (|m_0| + d - 1 - 2k) \frac{h_2a_2^2}{\Delta} f_{k+1} \quad (0 \leq k \leq d-1),$$

$$(4.3) \quad \partial_2 f_k = (2d - 2k + 1) \frac{h_1a_1^2}{\Delta} f_k - (|m_0| - d - 1 + 2k) \frac{h_1a_1^2}{\Delta} f_{k-1} \quad (1 \leq k \leq d),$$

$$(4.4) \quad [(\partial_1 + \partial_2)^2 + 2(\lambda_2 - 2)(\partial_1 + \partial_2) - 8\pi(h_1a_1^2\partial_1 + h_2a_2^2\partial_2) - 4(\lambda_2 - 1)] f_k(a) = 0 \quad (0 \leq k \leq d).$$

Here, $\Delta = h_1a_1^2 - h_2a_2^2$.

Proof. For $0 \leq k \leq d$, put

$$\check{f}_k(a) = \int_0^1 F(2\pi h_1a_1^2t + 2\pi h_2a_2^2(1-t)) t^{\frac{|m_0|-d-1}{2}+k} (1-t)^{\frac{|m_0|+d-1}{2}-k} dt.$$

Then we can verify that

$$\partial_1 f_k = (\sqrt{h_1}a_1)^{2k+1} (\sqrt{h_2}a_2)^{2d+1-2k} \Delta^{\frac{|m_0|-d}{2}} \left[\partial_1 + (2k+1) + (|m_0| - d) \frac{h_1a_1^2}{\Delta} \right] \check{f}_k$$

and

$$\partial_1 \check{f}_k = -(|m_0| - d + 1 + 2k) \frac{h_1 a_1^2}{\Delta} \check{f}_k + (|m_0| + d - 1 - 2k) \frac{h_1 a_1^2}{\Delta} \check{f}_{k+1}$$

for $0 \leq k \leq d-1$. Therefore, we obtain the formula (4.2). Similarly, we have

$$\partial_2 f_k = (\sqrt{h_1} a_1)^{2k+1} (\sqrt{h_2} a_2)^{2d+1-2k} \Delta^{\frac{|m_0|-d}{2}} \left[\partial_2 + (2d+1-2k) - (|m_0| - d) \frac{h_2 a_2^2}{\Delta} \right] \check{f}_k$$

and

$$\partial_2 \check{f}_k = -(|m_0| - d - 1 + 2k) \frac{h_2 a_2^2}{\Delta} \check{f}_{k-1} + (|m_0| + d + 1 - 2k) \frac{h_2 a_2^2}{\Delta} \check{f}_k$$

for $1 \leq k \leq d$. Thus, we have the formula (4.3).

Let us prove (4.4). Put Ω and $\check{\Omega}$ be the partial differential operators defined by

$$\Omega = (\partial_1 + \partial_2)^2 + 2(\lambda_2 - 2)(\partial_1 + \partial_2) - 8\pi(h_1 a_1^2 \partial_1 + h_2 a_2^2 \partial_2) - 4(\lambda_2 - 1)$$

and

$$\begin{aligned} \check{\Omega} = & (\partial_1 + \partial_2)^2 + 2(|m_0| + d + \lambda_2)(\partial_1 + \partial_2) - 8\pi(h_1 a_1^2 \partial_1 + h_2 a_2^2 \partial_2) \\ & - 8\pi(h_1 a_1^2 + h_2 a_2^2)(|m_0| + 1) - 8\pi(h_1 a_1^2 - h_2 a_2^2)(2k - d) + (|m_0| + d)(|m_0| + d + 2\lambda_2). \end{aligned}$$

Then we can check that

$$\Omega f_k = (\sqrt{h_1} a_1)^{2k+1} (\sqrt{h_2} a_2)^{2d+1-2k} \Delta^{\frac{|m_0|-d}{2}} \check{\Omega} \check{f}_k.$$

By interchanging differentiation and integration, we have

$$\check{\Omega} \check{f}_k(a) = \int_0^1 G(2\pi h_1 a_1^2 t + 2\pi h_2 a_2^2 (1-t)) t^{\frac{|m_0|-d-1}{2}+k} (1-t)^{\frac{|m_0|+d-1}{2}-k} dt$$

with

$$G(x) = 4 \left[x^2 \frac{d^2}{dx^2} - \{2x - (|m_0| + \lambda_1 + 1)\} x \frac{d}{dx} - 2(|m_0| + 1)x + \frac{(|m_0| + \lambda_1)^2 - \lambda_2^2}{4} \right] F(x).$$

Put $F(x) = e^x x^{-(|m_0|+\lambda_1+1)/2} H(x)$. Then we have

$$G(x) = 4e^x x^{-(|m_0|+\lambda_1+1)/2} \cdot x^2 \left[\frac{d^2}{dx^2} + \left\{ -1 + \frac{\lambda_1 - |m_0| - 1}{x} + \frac{1 - \lambda_2^2}{4x^2} \right\} \right] H(x).$$

It is known that the differential equation

$$\left[\frac{d^2}{dx^2} + \left\{ -1 + \frac{\lambda_1 - |m_0| - 1}{x} + \frac{1 - \lambda_2^2}{4x^2} \right\} \right] H(x) = 0$$

has two linearly independent solutions:

$$W_{\frac{\lambda_1 - |m_0| - 1}{2}, \frac{\lambda_2}{2}}(2x), \quad M_{\frac{\lambda_1 - |m_0| - 1}{2}, \frac{\lambda_2}{2}}(2x).$$

Here, $W_{\kappa, \mu}(z)$ and $M_{\kappa, \mu}(z)$ are Whittaker's confluent hypergeometric functions. (See [10] Chapter 16 for definition.) Therefore, we have (4.4). It completes the proof. \square

5. MAIN RESULTS

We state our main results on an explicit integral formula of Siegel-Whittaker functions which are of rapid decay for the large discrete series representations of $\mathrm{Sp}(2, \mathbb{R})$.

Theorem 5.1. *Let $\pi = \pi_\Lambda$ be a large discrete series representation of G with the Harish-Chandra parameter $\Lambda = (\lambda_1 - 1, \lambda_2) \in \Xi_\Pi$ and its minimal K -type $\tau = \tau_{(\lambda_1, \lambda_2)}$. Let ξ be a unitary character of N_S associated with a positive definite matrix $H_\xi = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$. Put $\eta = \chi_{m_0} \boxtimes \xi$, as in (2.1).*

We assume that

$$|m_0| \geq d \quad \text{and} \quad |m_0| \equiv d \pmod{2}.$$

Then we have the following:

- (i) $\dim_{\mathbb{C}} \mathrm{SW}(\pi, \eta, \tau)^{rap} = 1$.
- (ii) *Let $\{f_k(a)\}_{k=0}^d$ be the partially confluent hypergeometric functions, defined in Definition 4.1. We consider the following \mathbb{C} -linear combinations $\{g_j(a)\}_{j=0}^d$ of elements of $\{f_k(a)\}_{k=0}^d$, given by*

$$(5.1) \quad g_j(a) = \sum_{k=0}^d x_{jk} f_k(a),$$

where the complex numbers $\{x_{jk}\}_{0 \leq j, k \leq d}$ are given by

$$(5.2) \quad \begin{aligned} x_{jk} = & (-1)^{j+k} (\mathrm{sgn}(m_0) \sqrt{-1})^j \binom{d}{2k-j} |m_0|^{\delta(j)} \prod_{l=1}^k \frac{2l-1}{|m_0| - d + 2l - 1} \\ & \times \sum_{r=0}^{[j/2]} \binom{[j/2]}{r} \frac{(d-2r-\delta(j))!}{d!} \prod_{l=1}^r \left\{ \frac{2k-j-2l+2-\delta(j)}{2k-2l+1} (|m_0|^2 - (d-2l+2)^2) \right\} \\ & (\delta(j) = 1 \text{ if } j \text{ is odd, otherwise } 0). \end{aligned}$$

Then the function

$$(5.3) \quad \phi_{SW}(a) = \sum_{j=0}^d \left\{ (\sqrt{h_1} a_1)^{\lambda_1-j} (\sqrt{h_2} a_2)^{\lambda_2+j} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} g_j(a) \right\} v_j$$

gives a non-zero element in $\mathrm{SW}(\pi, \eta, \tau)^{rap}|_A$ which is unique up to constant multiple.

6. PROOF OF MAIN RESULTS

We claim that the coefficient functions $c_0(a), c_1(a), \dots, c_d(a)$ appearing in the Siegel-Whittaker function $\phi_{SW}(a)$, in Proposition 3.1, are \mathbb{C} -linear combination of the confluent hypergeometric functions $f_0(a), f_1(a), \dots, f_d(a)$ defined in Definition 4.1.

Proposition 6.1. *We assume that*

$$|m_0| \geq d \quad \text{and} \quad |m_0| \equiv d \pmod{2}.$$

Let $\{x_{jk}\}_{0 \leq j, k \leq d}$ be a sequence of complex numbers, which satisfy

$$(6.1) \quad j x_{j-1,k} + \sqrt{-1} m_0 x_{j,k} + (d - 2k + j + 1) x_{j+1,k} - (|m_0| - d + 2k + 1) x_{j+1,k+1} = 0$$

$$(0 \leq j \leq d-1, 0 \leq k \leq d),$$

$$(6.2) \quad (j - 2k - 1) x_{j-1,k} + (|m_0| + d - 2k + 1) x_{j-1,k-1} + \sqrt{-1} m_0 x_{j,k} - (d - j) x_{j+1,k} = 0$$

$$(1 \leq j \leq d, 0 \leq k \leq d),$$

$$(6.3) \quad x_{1,0} = x_{2,0} = \cdots = x_{d,0} = 0,$$

and

$$(6.4) \quad x_{0,d} = x_{1,d} = \cdots = x_{d-1,d} = 0.$$

For $0 \leq j \leq d$, define

$$g_j(a) = \sum_{k=0}^d x_{jk} f_k(a),$$

then $\{g_j(a)\}_{j=0}^d$ is a smooth solution of the system of the difference-differential equations (3.1), (3.2) and (3.3) in Proposition 3.1.

Proof. By using Proposition 4.2, we see that

$$\begin{aligned} & \frac{\Delta}{h_2 a_2^2} \left[\partial_1 + j \frac{h_2 a_2^2}{\Delta} \right] g_{j-1}(a) \\ &= \sum_{k=0}^d (j - 2k - 1) x_{j-1,k} f_k + \sum_{k=0}^d (|m_0| + d - 2k - 1) x_{j-1,k} f_{k+1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{\Delta}{h_2 a_2^2} \left[\partial_1 + j \frac{h_2 a_2^2}{\Delta} \right] g_{j-1}(a) + \sqrt{-1} m_0 g_j(a) - (d - j) g_{j+1}(a) \\ &= \sum_{k=0}^d (j - 2k - 1) x_{j-1,k} f_k + \sum_{k=1}^d (|m_0| + d - 2k + 1) x_{j-1,k-1} f_k \\ & \quad + \sqrt{-1} m_0 \sum_{k=0}^d x_{j,k} f_k - (d - j) \sum_{k=0}^d x_{j+1,k} f_k \\ &= 0 \quad (1 \leq j \leq d). \end{aligned}$$

This is the desired (3.1) for $\{g_j(a)\}_{j=0}^d$. In the above calculation, we used the relations (6.2) and (6.4) on $\{x_{jk}\}_{0 \leq j, k \leq d}$. Again, by using Proposition 4.2, we see that

$$\begin{aligned} & \frac{\Delta}{h_1 a_1^2} \left[\partial_2 - (d - j) \frac{h_1 a_1^2}{\Delta} \right] g_{j+1}(a) \\ &= \sum_{k=0}^d (d - 2k + j + 1) x_{j+1,k} f_k - \sum_{k=0}^d (|m_0| - d + 2k - 1) x_{j+1,k} f_{k-1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
& jg_{j-1}(a) + \sqrt{-1}m_0g_j(a) + \frac{\Delta}{h_1a_1^2} \left[\partial_2 - (d-j)\frac{h_1a_1^2}{\Delta} \right] g_{j+1}(a) \\
&= j \sum_{k=0}^d x_{j-1,k} f_k + \sqrt{-1}m_0 \sum_{k=0}^d x_{j,k} f_k + \sum_{k=0}^d (d-2k+j+1)x_{j+1,k} f_k \\
&\quad - \sum_{k=0}^{d-1} (|m_0| - d + 2k + 1)x_{j+1,k+1} f_k \\
&= 0 \quad (0 \leq j \leq d-1).
\end{aligned}$$

This is the desired (3.2) for $\{g_j(a)\}_{j=0}^d$. In the above calculation, we used the relations (6.1) and (6.3) on $\{x_{jk}\}_{0 \leq j,k \leq d}$.

By considering $(3.1) \times h_1a_1^2 + (3.2) \times h_2a_2^2 + (3.3)$, we have

$$\begin{aligned}
(6.5) \quad & h_1a_1^2[(\partial_1 + \partial_2) - 8\pi h_2a_2^2 + 2\lambda_2 - 2]c_{j-1}(a) \\
& + h_2a_2^2[(\partial_1 + \partial_2) - 8\pi h_1a_1^2 + 2\lambda_2 - 2]c_{j+1}(a) = 0 \quad (1 \leq j \leq d-1).
\end{aligned}$$

By considering $(3.1) \times h_1a_1^2 - (3.2) \times h_2a_2^2$, we have

$$(6.6) \quad h_1a_1^2\partial_1 c_{j-1}(a) - h_2a_2^2\partial_2 c_{j+1}(a) = 0 \quad (1 \leq j \leq d-1).$$

Operating $\partial_2(h_2a_2^2)^{-1}$ on both hand sides of (6.5), we have

$$\begin{aligned}
(6.7) \quad & \partial_2 \frac{h_1a_1^2}{h_2a_2^2} [(\partial_1 + \partial_2) - 8\pi h_2a_2^2 + 2\lambda_2 - 2]c_{j-1}(a) \\
& + [(\partial_1 + \partial_2) - 8\pi h_1a_1^2 + 2\lambda_2 - 2]\partial_2 c_{j+1}(a) = 0 \quad (1 \leq j \leq d-1).
\end{aligned}$$

Combining (6.7) and (6.6), we have, for $0 \leq j \leq d-2$,

$$(6.8) \quad [(\partial_1 + \partial_2)^2 + 2(\lambda_2 - 2)(\partial_1 + \partial_2)\partial_2 - 8\pi(h_1a_1^2\partial_1 + h_2a_2^2\partial_2) - 4(\lambda_2 - 1)]c_j(a) = 0.$$

Operating $\partial_1(h_1a_1^2)^{-1}$ on both hand sides of (6.5) and combining with (6.6), we have (6.8) for $2 \leq j \leq d$. As a result, we have (6.8) for $0 \leq j \leq d$.

Lastly we prove that $\{g_j(a)\}_{j=0}^d$ satisfy (3.3). By Proposition 4.2, $f_0(a), f_1(a), \dots, f_d(a)$ satisfy the same differential equation (6.8), therefore their \mathbb{C} -linear combinations $\{g_j(a)\}_{j=0}^d$ also satisfy (6.8). Since $\{g_j(a)\}_{j=0}^d$ satisfy (6.6) and (6.8), we see that

$$(6.9) \quad \partial_2 \frac{F_j(a)}{h_2a_2^2} = \partial_1 \frac{F_j(a)}{h_1a_1^2} = 0 \quad (1 \leq j \leq d-1),$$

where we put

$$\begin{aligned}
F_j(a) = & h_1a_1^2[(\partial_1 + \partial_2) - 8\pi h_2a_2^2 + 2\lambda_2 - 2]g_{j-1}(a) \\
& + h_2a_2^2[(\partial_1 + \partial_2) - 8\pi h_1a_1^2 + 2\lambda_2 - 2]g_{j+1}(a).
\end{aligned}$$

By (6.9), there exist constants β_j ($1 \leq j \leq d-1$) such that

$$F_j(a) = \beta_j h_1a_1^2 h_2a_2^2.$$

Let us determine β_j . For $y > 0$, define $L = \{(a_1, a_2) \in A \mid h_1 a_1^2 = h_2 a_2^2 = y\}$. We show that $(F_j|_L)(y) = 0$ to deduce $\beta_j = 0$. We can verify that

$$\begin{aligned} (\partial_1 + \partial_2) f_k &= (\sqrt{h_1} a_1)^{2k+1} (\sqrt{h_2} a_2)^{2d+1-2k} \Delta^{\frac{|m_0|-d}{2}} [(\partial_1 + \partial_2) + (d + |m_0| + 2)] \\ &\quad \times \int_0^1 F(2\pi h_1 a_1^2 t + 2\pi h_2 a_2^2 (1-t)) t^{\frac{|m_0|-d-1}{2}+k} (1-t)^{\frac{|m_0|+d-1}{2}-k} dt \\ &= (\sqrt{h_1} a_1)^{2k+1} (\sqrt{h_2} a_2)^{2d+1-2k} \Delta^{\frac{|m_0|-d}{2}} \\ &\quad \times \int_0^1 F^*(2\pi h_1 a_1^2 t + 2\pi h_2 a_2^2 (1-t)) t^{\frac{|m_0|-d-1}{2}+k} (1-t)^{\frac{|m_0|+d-1}{2}-k} dt. \end{aligned}$$

Here,

$$F^*(x) = \left(\left(2x \frac{d}{dx} + (d + |m_0| + 2) \right) F \right)(x).$$

There are two cases:

- (i) If $|m_0| - d \geq 2$, then both $(\partial_1 + \partial_2) f_k$ and f_k have zeros at $\{h_1 a_1^2 - h_2 a_2^2 = 0\}$. Hence, both $(\partial_1 + \partial_2) g_j$ and g_j have zeros at there. Therefore we have $(F_j|_L)(y) = 0$.
- (ii) Suppose that $|m_0| = d$, then we have

$$((\partial_1 + \partial_2) f_k)|_L(y) = y^{d+1} F^*(2\pi y) B\left(k + \frac{1}{2}, d - k + \frac{1}{2}\right).$$

Here,

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

is the Beta function. Then we can verify that

$$\begin{aligned} ((\partial_1 + \partial_2) g_j)|_L(y) &= \left(\sum_{k=0}^d x_{jk} B\left(k + \frac{1}{2}, d - k + \frac{1}{2}\right) \right) y^{d+1} F^*(2\pi y) \\ (6.10) \quad &= (\text{sgn}(m_0) \sqrt{-1})^j 2^{-d} \pi y^{d+1} F^*(2\pi y). \end{aligned}$$

To derive (6.10), we used (6.16) in Proposition 6.2: (We will prove later.)

$$x_{jk} = (-1)^{j+k} (\text{sgn}(m_0) \sqrt{-1})^j \binom{d}{2k-j} \quad (\text{when } |m_0| = d),$$

under the condition $x_{0,0} = 1$, and the equality:

$$(6.11) \quad \sum_{k=0}^d (-1)^k \binom{2k}{j} \binom{2d-2k}{d-j} \binom{d}{k} = (-1)^j 2^d \binom{d}{j}.$$

Let $S_{d,j}$ be the left hand side of (6.11). We remark that the above equality (6.11) is proved by showing that

$$S_{d,j} = -\frac{2d}{j} S_{d-1,j-1} \quad (j \geq 1) \text{ and } S_{d,0} = 2^d.$$

See p.620 no. 63 in [11] for the equality $S_{d,0} = 2^d$.

Similarly, we also obtain

$$g_j|_L(y) = (\mathrm{sgn}(m_0)\sqrt{-1})^j 2^{-d} \pi y^{d+1} F(2\pi y).$$

Therefore, we have

$$((\partial_1 + \partial_2)(g_{j-1} + g_{j+1}))|_L(y) = (g_{j-1} + g_{j+1})|_L(y) = 0,$$

for $1 \leq j \leq d-1$. Therefore we have $(F_j|_L)(y) = 0$.

In any cases, $\{g_j(a)\}_{j=0}^d$ satisfy (6.5). Hence, $\{g_j(a)\}_{j=0}^d$ satisfy (3.3) and it completes the proof. \square

Proposition 6.2. *We assume that*

$$|m_0| \geq d \quad \text{and} \quad |m_0| \equiv d \pmod{2}.$$

Let $\{x_{jk}\}_{0 \leq j, k \leq d}$ be a sequence of complex numbers, which satisfy

$$(6.12) \quad j x_{j-1,k} + \sqrt{-1} m_0 x_{j,k} + (d - 2k + j + 1) x_{j+1,k} - (|m_0| - d + 2k + 1) x_{j+1,k+1} = 0$$

$(0 \leq j \leq d-1, 0 \leq k \leq d),$

$$(6.13) \quad (j - 2k - 1) x_{j-1,k} + (|m_0| + d - 2k + 1) x_{j-1,k-1} + \sqrt{-1} m_0 x_{j,k} - (d - j) x_{j+1,k} = 0$$

$(1 \leq j \leq d, 0 \leq k \leq d),$

$$(6.14) \quad x_{1,0} = x_{2,0} = \cdots = x_{d,0} = 0,$$

and

$$(6.15) \quad x_{0,d} = x_{1,d} = \cdots = x_{d-1,d} = 0.$$

Then the sequence $\{x_{jk}\}_{0 \leq j, k \leq d}$ is uniquely determined and given by, up to a constant multiple,

$$(6.16) \quad x_{jk} = (-1)^{j+k} (\mathrm{sgn}(m_0)\sqrt{-1})^j \binom{d}{2k-j} |m_0|^{\delta(j)} \prod_{l=1}^k \frac{2l-1}{|m_0| - d + 2l - 1}$$

$$\times \sum_{r=0}^{\lfloor j/2 \rfloor} \binom{\lfloor j/2 \rfloor}{r} \frac{(d - 2r - \delta(j))!}{d!} \prod_{l=1}^r \left\{ \frac{2k - j - 2l + 2 - \delta(j)}{2k - 2l + 1} (|m_0|^2 - (d - 2l + 2)^2) \right\},$$

where, $\delta(j) = 1$ if j is odd, otherwise 0.

Proof. We may assume that $x_{0,0} = 1$. Let us write down (6.12) with $j = 0, 1$ and k replaced by $k-1, k$, and (6.13) with $j = 1$ and k replaced by $k-1, k, k+1$. Then we have the

following system of seven difference equations:

$$\begin{aligned}
(6.17) \quad & \sqrt{-1}m_0 x_{0,k-1} + (d - 2k + 3) x_{1,k-1} - (|m_0| - d + 2k - 1) x_{1,k} = 0, \\
& \sqrt{-1}m_0 x_{0,k} + (d - 2k + 1) x_{1,k} - (|m_0| - d + 2k + 1) x_{1,k+1} = 0, \\
& x_{0,k-1} + \sqrt{-1}m_0 x_{1,k-1} + (d - 2k + 4) x_{2,k-1} - (|m_0| - d + 2k - 1) x_{2,k} = 0, \\
& x_{0,k} + \sqrt{-1}m_0 x_{1,k} + (d - 2k + 2) x_{2,k} - (|m_0| - d + 2k + 1) x_{2,k+1} = 0, \\
& - (2k - 2) x_{0,k-1} + (|m_0| + d - 2k + 3) x_{0,k-2} + \sqrt{-1}m_0 x_{1,k-1} - (d - 1) x_{2,k-1} = 0, \\
& - 2k x_{0,k} + (|m_0| + d - 2k + 1) x_{0,k-1} + \sqrt{-1}m_0 x_{1,k} - (d - 1) x_{2,k} = 0, \\
& - (2k + 2) x_{0,k+1} + (|m_0| + d - 2k - 1) x_{0,k} + \sqrt{-1}m_0 x_{1,k+1} - (d - 1) x_{2,k+1} = 0.
\end{aligned}$$

We eliminate $x_{j,k}$, $x_{j,k\pm 1}$ ($j = 1, 2$) in the above system. Then we have

$$\begin{aligned}
(6.18) \quad & - 2(k + 1)(|m_0| - d + 2k + 1)(|m_0| - d + 2k - 1) x_{0,k+1} \\
& + \{2k(4d - 6k + 3) - d(d - 1)\}(|m_0| - d + 2k - 1) x_{0,k} \\
& + \left[2(d - 2k + 2)\{d(d - 1) - (2k - 2)(2d - 2k + 3)\} \right. \\
& \quad \left. + (2k - 2)(|m_0| + d - 2k + 1)(|m_0| - d + 2k - 1) \right] x_{0,k-1} \\
& + (d - 2k + 4)(d - 2k + 3)(|m_0| + d - 2k + 3) x_{0,k-2} \\
& = 0.
\end{aligned}$$

By solving the difference equation (6.18) for $\{x_{0,k}\}$ with $x_{0,0} = 1$, we obtain

$$(6.19) \quad x_{0,k} = (-1)^k \binom{d}{2k} \prod_{l=1}^k \frac{2l - 1}{|m_0| - d + 2l - 1}.$$

From the second equation of (6.17), (6.19) and (6.14), we have the following difference equation for $\{x_{1,k}\}$:

$$(|m_0| - d + 2k + 1) x_{1,k+1} = (d - 2k + 1) x_{1,k} + \sqrt{-1}m_0 (-1)^k \binom{d}{2k} \prod_{l=1}^k \frac{2l - 1}{|m_0| - d + 2l - 1}$$

with $x_{1,0} = 0$. Thus, we obtain

$$(6.20) \quad x_{1,k} = (-1)^{k+1} (\operatorname{sgn}(m_0) \sqrt{-1}) \binom{d}{2k-1} \frac{|m_0|}{d} \prod_{l=1}^k \frac{2l - 1}{|m_0| - d + 2l - 1}.$$

We prove the formula (6.16) for general x_{jk} by induction on the index j . For $0 \leq 2p, 2p+1, k \leq d$, let us prove

$$(6.21) \quad \begin{aligned} x_{2p,k} = & (-1)^{k+p} \binom{d}{2k-2p} \prod_{l=1}^k \frac{2l-1}{|m_0| - d + 2l - 1} \\ & \times \sum_{r=0}^p \binom{p}{r} \frac{(d-2r)!}{d!} \prod_{l=1}^r \left\{ \frac{2k-2p-2l+2}{2k-2l+1} (|m_0|^2 - (d-2l+2)^2) \right\}, \end{aligned}$$

$$(6.22) \quad \begin{aligned} x_{2p+1,k} = & (-1)^{k+p+1} (\mathrm{sgn}(m_0) \sqrt{-1}) \binom{d}{2k-2p-1} |m_0| \prod_{l=1}^k \frac{2l-1}{|m_0| - d + 2l - 1} \\ & \times \sum_{r=0}^p \binom{p}{r} \frac{(d-2r-1)!}{d!} \prod_{l=1}^r \left\{ \frac{2k-2p-2l}{2k-2l+1} (|m_0|^2 - (d-2l+2)^2) \right\}. \end{aligned}$$

Suppose that $2p+1 \leq d$ and the above formulas are true for $x_{2p-1,k}, x_{2p,k}$ ($0 \leq k \leq d$), then substitute them into (6.13):

$$(6.23) \quad x_{2p+1,k} = \frac{2p-2k-1}{d-2p} x_{2p-1,k} + \frac{|m_0| + d - 2k + 1}{d-2p} x_{2p-1,k-1} + \frac{\sqrt{-1} m_0}{d-2p} x_{2p,k}.$$

Put

$$a(k) = \prod_{l=1}^k \frac{2l-1}{|m_0| - d + 2l - 1}, \quad b(r; k, p) = \prod_{l=1}^r \left\{ \frac{2k-2p-2l}{2k-2l+1} (|m_0|^2 - (d-2l+2)^2) \right\}.$$

Then, we have

$$(6.24) \quad \begin{aligned} & \frac{(-1)^{k+p+1} (2p-2k-1)}{\sqrt{-1} m_0 (d-2p)} x_{2p-1,k} \\ &= -\frac{2p-2k-1}{d-2p} \binom{d}{2k-2p+1} a(k) \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d-2r-1)!}{d!} b(r; k, p-1) \\ &= \frac{a(k)}{d-2p} \binom{d}{2k-2p-1} (d-2k+2p+1)(d-2k+2p) \\ & \quad \times \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d-2r-1)!}{d!} \frac{b(r; k, p)}{2k-2p-2r}, \end{aligned}$$

and

(6.25)

$$\begin{aligned}
& \frac{(-1)^{k+p+1}(|m_0| + d - 2k + 1)}{\sqrt{-1} m_0(d - 2p)} x_{2p-1, k-1} \\
&= \frac{|m_0| + d - 2k + 1}{d - 2p} \binom{d}{2k - 2p - 1} a(k - 1) \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} b(r; k - 1, p - 1) \\
&= \frac{a(k)}{d - 2p} \binom{d}{2k - 2p - 1} \{ |m_0|^2 - (d - 2k + 1)^2 \} \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} \frac{b(r; k, p)}{2k - 2r - 1} \\
&= \frac{a(k)}{d - 2p} \binom{d}{2k - 2p - 1} \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} \frac{b(r + 1; k, p)}{2k - 2p - 2r - 2} \\
&\quad + \frac{a(k)}{d - 2p} \binom{d}{2k - 2p - 1} \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} (2d - 2r - 2k + 1) b(r; k, p).
\end{aligned}$$

Here, we used the equality:

$$\{ |m_0|^2 - (d - 2k + 1)^2 \} = \{ |m_0|^2 - (d - 2r)^2 \} + (2d - 2r - 2k + 1)(2k - 2r - 1)$$

in the above calculation. By noting the equality:

$$\frac{(d - 2k + 2p + 1)(d - 2k + 2p)}{2k - 2p - 2r} + (2d - 2r - 2k + 1) = \frac{(d - 2r)(d - 2r + 1)}{2k - 2p - 2r} - 2p,$$

we have

(6.24)+(6.25)

$$\begin{aligned}
&= \frac{a(k)}{d - 2p} \binom{d}{2k - 2p - 1} \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} \left\{ \frac{(d - 2r)(d - 2r + 1)}{2k - 2p - 2r} - 2p \right\} b(r; k, p) \\
&\quad + \frac{a(k)}{d - 2p} \binom{d}{2k - 2p - 1} \sum_{r=1}^p \binom{p}{r-1} \frac{(d - 2r + 1)!}{d!} \frac{b(r; k, p)}{2k - 2p - 2r} \\
&= \frac{a(k)}{d - 2p} \binom{d}{2k - 2p - 1} \sum_{r=0}^p \binom{p}{r} \frac{(d - 2r + 1)!}{d!} \frac{b(r; k, p)}{2k - 2p - 2r} \\
&\quad - 2p \frac{a(k)}{d - 2p} \binom{d}{2k - 2p - 1} \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d - 2r - 1)!}{d!} b(r; k, p).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 (6.26) \quad & \frac{(-1)^{k+p+1}}{\sqrt{-1} m_0(d-2p)} x_{2p,k} \\
 &= -\frac{1}{d-2p} \binom{d}{2k-2p} a(k) \sum_{r=0}^p \binom{p}{r} \frac{(d-2r)!}{d!} b(r; k, p-1) \\
 &= -\frac{a(k)}{d-2p} \binom{d}{2k-2p-1} (d-2k+2p+1) \sum_{r=0}^p \binom{p}{r} \frac{(d-2r)!}{d!} \frac{b(r; k, p)}{2k-2p-2r} \\
 &= \frac{a(k)}{d-2p} \binom{d}{2k-2p-1} \sum_{r=0}^p \binom{p}{r} \frac{(d-2r)!}{d!} b(r; k, p) \\
 &\quad - \frac{a(k)}{d-2p} \binom{d}{2k-2p-1} \sum_{r=0}^p \binom{p}{r} \frac{(d-2r+1)!}{d!} \frac{b(r; k, p)}{2k-2p-2r}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 & (6.24) + (6.25) + (6.26) \\
 &= \frac{a(k)}{d-2p} \binom{d}{2k-2p-1} \sum_{r=0}^p \binom{p}{r} \frac{(d-2r)!}{d!} b(r; k, p) \\
 &\quad - 2p \frac{a(k)}{d-2p} \binom{d}{2k-2p-1} \sum_{r=0}^{p-1} \binom{p-1}{r} \frac{(d-2r-1)!}{d!} b(r; k, p) \\
 &= a(k) \binom{d}{2k-2p-1} \sum_{r=0}^p \binom{p}{r} \frac{(d-2r-1)!}{d!} b(r; k, p).
 \end{aligned}$$

By (6.23) and the above formula, we obtain

$$x_{2p+1,k} = (-1)^{k+p+1} (m_0 \sqrt{-1}) a(k) \binom{d}{2k-2p-1} \sum_{r=0}^p \binom{p}{r} \frac{(d-2r-1)!}{d!} b(r; k, p).$$

Therefore, the formula is valid for $x_{2p+1,k}$ ($0 \leq k \leq d$). Similarly, if we suppose that $2p \leq d$ and the formulas (6.21) and (6.22) are true for $x_{2p-1,k}$, $x_{2p-2,k}$ ($0 \leq k \leq d$), then we can prove the formula is also valid for $x_{2p,k}$ ($0 \leq k \leq d$). It completes the proof. \square

Let us complete the proof of Theorem 5.1.

Proof of 5.1. By the condition $|m_0| \geq d$, $|m_0| \equiv d \pmod{2}$, Propositions 6.1 and 6.2, $\{g_j(a)\}_{j=0}^d$ is a *non-zero* smooth solution of the system (3.1), (3.2) and (3.3) in Proposition 3.1. Furthermore, we can check that all of $\{e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} g_j(a)\}_{j=0}^d$ are rapidly decreasing when each a_1, a_2 tends to infinity, by Definition 4.1. We completes the proof. \square

7. REMARKS ON $\{x_{jk}\}$

We have some remarks on the sequence of complex numbers $\{x_{jk}\}_{0 \leq j, k \leq d}$ appearing in Theorem 5.1 and Proposition 6.2. Let $X_d = (x_{jk})_{0 \leq j, k \leq d} \in M_{d+1}(\mathbb{C})$ be the square matrix of size $(d+1)$, defined by the sequence $\{x_{jk}\}_{0 \leq j, k \leq d}$.

Put $\varepsilon = \operatorname{sgn}(m_0)\sqrt{-1}$ and $t = |m_0|$. Then x_{jk} is given by

(7.1)

$$x_{jk} = (-1)^{j+k} \varepsilon^j \binom{d}{2k-j} t^{\delta(j)} \prod_{l=1}^k \frac{2l-1}{t-d+2l-1} \\ \times \sum_{r=0}^{[j/2]} \binom{[j/2]}{r} \frac{(d-2r-\delta(j))!}{d!} \prod_{l=1}^r \left\{ \frac{2k-j-2l+2-\delta(j)}{2k-2l+1} (t^2 - (d-2l+2)^2) \right\},$$

where, $\delta(j) = 1$ if j is odd, otherwise 0.

Furthermore, we set

$$(7.2) \quad Z_d = (z_{jk})_{0 \leq j, k \leq d} \in M_{d+1}(\mathbb{C}) \text{ with } z_{jk} = \varepsilon^{-j} x_{jk}.$$

Though the Harish-Chandra parameter $\Lambda = (\lambda_1 - 1, \lambda_2) \in \Xi_{\Pi}$ implies that $d = \lambda_1 - \lambda_2 \geq 4$, we formally write down matrices Z_d for $1 \leq d \leq 7$.

Example 7.1 (Z_d for $1 \leq d \leq 7$).

$$Z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 1 & -\frac{1}{t-1} & 0 \\ 0 & \frac{t}{t-1} & 0 \\ 0 & -\frac{1}{t-1} & 1 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 1 & -\frac{3}{t-2} & 0 & 0 \\ 0 & \frac{t}{t-2} & -\frac{1}{t-2} & 0 \\ 0 & -\frac{1}{t-2} & \frac{t}{t-2} & 0 \\ 0 & 0 & -\frac{3}{t-2} & 1 \end{bmatrix},$$

$$Z_4 = \begin{bmatrix} 1 & -\frac{6}{t-3} & \frac{3}{(t-3)(t-1)} & 0 & 0 \\ 0 & \frac{t}{t-3} & -\frac{3t}{(t-3)(t-1)} & 0 & 0 \\ 0 & -\frac{1}{t-3} & \frac{t^2+2}{(t-3)(t-1)} & -\frac{1}{t-3} & 0 \\ 0 & 0 & -\frac{3t}{(t-3)(t-1)} & \frac{t}{t-3} & 0 \\ 0 & 0 & \frac{3}{(t-3)(t-1)} & -\frac{6}{t-3} & 1 \end{bmatrix},$$

$$Z_5 = \begin{bmatrix} 1 & -\frac{10}{t-4} & \frac{15}{(t-4)(t-2)} & 0 & 0 & 0 \\ 0 & \frac{t}{t-4} & -\frac{6t}{(t-4)(t-2)} & \frac{3}{(t-4)(t-2)} & 0 & 0 \\ 0 & -\frac{1}{t-4} & \frac{t^2+5}{(t-4)(t-2)} & -\frac{3t}{(t-4)(t-2)} & 0 & 0 \\ 0 & 0 & -\frac{3t}{(t-4)(t-2)} & \frac{t^2+5}{(t-4)(t-2)} & -\frac{1}{t-4} & 0 \\ 0 & 0 & \frac{3}{(t-4)(t-2)} & -\frac{6t}{(t-4)(t-2)} & \frac{t}{t-4} & 0 \\ 0 & 0 & 0 & \frac{15}{(t-4)(t-2)} & -\frac{10}{t-4} & 1 \end{bmatrix},$$

$$Z_6 = \begin{bmatrix} 1 & -\frac{15}{t-5} & \frac{45}{(t-5)(t-3)} & -\frac{15}{(t-5)(t-3)(t-1)} & 0 & 0 & 0 \\ 0 & \frac{t}{t-5} & -\frac{10t}{(t-5)(t-3)} & \frac{15t}{(t-5)(t-3)(t-1)} & 0 & 0 & 0 \\ 0 & -\frac{1}{t-5} & \frac{t^2+9}{(t-5)(t-3)} & -\frac{3(2t^2+3)}{(t-5)(t-3)(t-1)} & \frac{3}{(t-5)(t-3)} & 0 & 0 \\ 0 & 0 & -\frac{3t}{(t-5)(t-3)} & \frac{t(t^2+14)}{(t-5)(t-3)(t-1)} & -\frac{3t}{(t-5)(t-3)} & 0 & 0 \\ 0 & 0 & \frac{3}{(t-5)(t-3)} & -\frac{3(2t^2+3)}{(t-5)(t-3)(t-1)} & \frac{t^2+9}{(t-5)(t-3)} & -\frac{1}{t-5} & 0 \\ 0 & 0 & 0 & \frac{15t}{(t-5)(t-3)(t-1)} & -\frac{10t}{(t-5)(t-3)} & \frac{t}{t-5} & 0 \\ 0 & 0 & 0 & -\frac{15}{(t-5)(t-3)(t-1)} & \frac{45}{(t-5)(t-3)} & -\frac{15}{t-5} & 1 \end{bmatrix},$$

$$Z_7 = \begin{bmatrix} 1 & -\frac{21}{t-6} & \frac{105}{(t-6)(t-4)} & -\frac{105}{(t-6)(t-4)(t-2)} & 0 & 0 & 0 & 0 \\ 0 & \frac{t}{t-6} & -\frac{15t}{(t-6)(t-4)} & \frac{45t}{(t-6)(t-4)(t-2)} & -\frac{15}{(t-6)(t-4)(t-2)} & 0 & 0 & 0 \\ 0 & -\frac{1}{t-6} & \frac{t^2+14}{(t-6)(t-4)} & -\frac{5(2t^2+7)}{(t-6)(t-4)(t-2)} & \frac{15t}{(t-6)(t-4)(t-2)} & 0 & 0 & 0 \\ 0 & 0 & -\frac{3t}{(t-6)(t-4)} & \frac{t(t^2+26)}{(t-6)(t-4)(t-2)} & -\frac{3(2t^2+7)}{(t-6)(t-4)(t-2)} & \frac{3}{(t-6)(t-4)} & 0 & 0 \\ 0 & 0 & \frac{3}{(t-6)(t-4)} & -\frac{3(2t^2+7)}{(t-6)(t-4)(t-2)} & \frac{t(t^2+26)}{(t-6)(t-4)(t-2)} & -\frac{3t}{(t-6)(t-4)} & 0 & 0 \\ 0 & 0 & 0 & \frac{15t}{(t-6)(t-4)(t-2)} & -\frac{5(2t^2+7)}{(t-6)(t-4)(t-2)} & \frac{t^2+14}{(t-6)(t-4)} & -\frac{1}{t-6} & 0 \\ 0 & 0 & 0 & -\frac{15}{(t-6)(t-4)(t-2)} & \frac{45t}{(t-6)(t-4)(t-2)} & -\frac{15t}{(t-6)(t-4)} & \frac{t}{t-6} & 0 \\ 0 & 0 & 0 & 0 & -\frac{105}{(t-6)(t-4)(t-2)} & \frac{105}{(t-6)(t-4)} & -\frac{21}{t-6} & 1 \end{bmatrix}.$$

By direct calculation, we have

Proposition 7.2 ($\det Z_d$ for $1 \leq d \leq 7$).

$$\det Z_1 = 1, \quad \det Z_2 = \frac{t}{t-1}, \quad \det Z_3 = \frac{t^2-1}{(t-2)^2}, \quad \det Z_4 = \frac{t^2(t^2-4)}{(t-1)^1(t-3)^3},$$

$$\det Z_5 = \frac{(t^2-1)^2(t^2-9)^1}{(t-2)^2(t-4)^4}, \quad \det Z_6 = \frac{t^3(t^2-4)^2(t^2-16)^1}{(t-1)^1(t-3)^3(t-5)^5},$$

$$\det Z_7 = \frac{(t^2-1)^3(t^2-9)^2(t^2-25)^1}{(t-2)^2(t-4)^4(t-6)^6}.$$

Since $\det X_d = \varepsilon^{d(d+1)/2} \det Z_d$, we have the following conjecture on $\det X_d$.

Conjecture 7.3. For a natural number q , we conjecture that

$$\det X_{2q-1} = (\mathrm{sgn}(m_0)\sqrt{-1})^{q(2q-1)} \frac{\prod_{l=1}^{q-1} (t^2 - (2l-1)^2)^{q-l}}{\prod_{l=1}^{q-1} (t-2l)^{2l}},$$

$$\det X_{2q} = (\mathrm{sgn}(m_0)\sqrt{-1})^{q(2q+1)} \frac{t^q \prod_{l=1}^{q-1} (t^2 - (2l)^2)^{q-l}}{\prod_{l=1}^q (t-2l+1)^{2l-1}}.$$

In this article, we have shown that the Siegel-Whittaker functions of rapid decay for the large discrete series representations of $\mathrm{Sp}(2, \mathbb{R})$ are described by the partially confluent hypergeometric functions $\{f_k(a)\}_{k=0}^d$. Conversely, if the above conjecture is true, we have,

Corollary 7.4. *We assume that*

$$|m_0| \geq d \quad \text{and} \quad |m_0| \equiv d \pmod{2}.$$

Let $\{c_j(a)\}_{j=0}^d$ be a smooth solution of the system (3.1), (3.2) and (3.3) in Proposition 3.1, and suppose that all of $\{e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} c_j(a)\}_{j=0}^d$ are rapidly decreasing.

If Conjecture 7.3 is true, then all of $\{f_k(a)\}_{k=0}^d$ are \mathbb{C} -linear combinations of elements of $\{c_j(a)\}_{j=0}^d$.

Proof. Since $|m_0| \geq d$ and $|m_0| \equiv d \pmod{2}$, we can check that

$$\det X_d \neq 0,$$

and X_d^{-1} exists. It completes the proof. \square

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